Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 18: Basic Counting Principles. Section 6.1

1 Basic Counting Principles

We cover first the two basic counting principles, the product rule and the sum rule.

Definition 1. (THE PRODUCT RULE) Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

Example 2. Let A be a set with 3 elements and B a set with 5 elements. How many functions are there from A to B? Generalize: How many functions are there from a set with m elements to a set with n elements?
Ans: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdots n = m^n$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from our set A with three elements to our set B with five elements.

Example 3. How many one-to-one functions are there from a set with m elements to one with n elements?
Ans: The first thing to note is that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n element. For $m \leq n$ we have

$$n(n-1)(n-2)\dots(n-m+1)$$

one-to-one functions from a set with m elements to one with n elements. In particular, there are only $5(4)(3) = 60$ different one-to-one functions from our set A with three elements to our set B with five elements.

Example 4. How many different bit strings of length n can we form? How many subsets does a finite set has?

Remark 5. The product rule is often phrased in terms of sets in this way: If $A_1, A_2, \dots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

As for the other rule, the sum rule:

Definition 6. (THE SUM RULE) If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 7. Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?
Ans: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Remark 8. The sum rule can be phrased in terms of sets as: If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \qquad \text{for} \quad A_i \cap A_j = \emptyset \quad i \neq j.$$